THE MBX CHALLENGE COMPETITION: A NEUTRON MATTER MODEL

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In this paper I report my solution to MBX Challenge Competition. Namely, the Bertsch, nonparametric model of neutron matter is analyzed and strong indications are found that, in the infinite system limit, the ground state is a Fermi liquid with an effective mass.

1 Introduction

As a challenge to the participants of the Tenth International Conference on Recent Progress in Many-Body Theory, G. F. Bertsch[1] proposed the following problem. It is:

What are the ground state properties of the many-body system composed of spin-1/2 Fermions interacting via a zero-range, infinite scattering-length contact interaction.

It may be assumed that the interaction has no two-body bound states. Also, the zero range is approached with finite-ranged forces and finite particle number by first taking the range to zero and then the particle number to infinity.

This problem is tricky in the following sense, if one reverses the limit order and takes the particle number to infinity before the range goes to zero, one obtains the well-known nuclear collapse result where the whole system collapses into a region of the order of the range of the potential in size. Likewise, if the particles were Bosons, collapse would occur. A fuller exposition of the solution may be found in reference 2.

2 Methods

How shall we solve this problem?

We will use a combination of two types of series expansions.

1. An expansion of the ground state energy in powers of the potential strength for fixed density.

2. A low density expansion of the ground state energy for fixed potential strength.

For ease of exposition, I will use the square-well potential,

$$V(r) = \begin{cases} -V_0, & \text{if } r < c, \\ 0, & \text{if } r > c. \end{cases}$$

For this potential, the strength is

$$s = \frac{4}{\pi^2}\frac{MV_0}{\hbar^2}c^2.$$

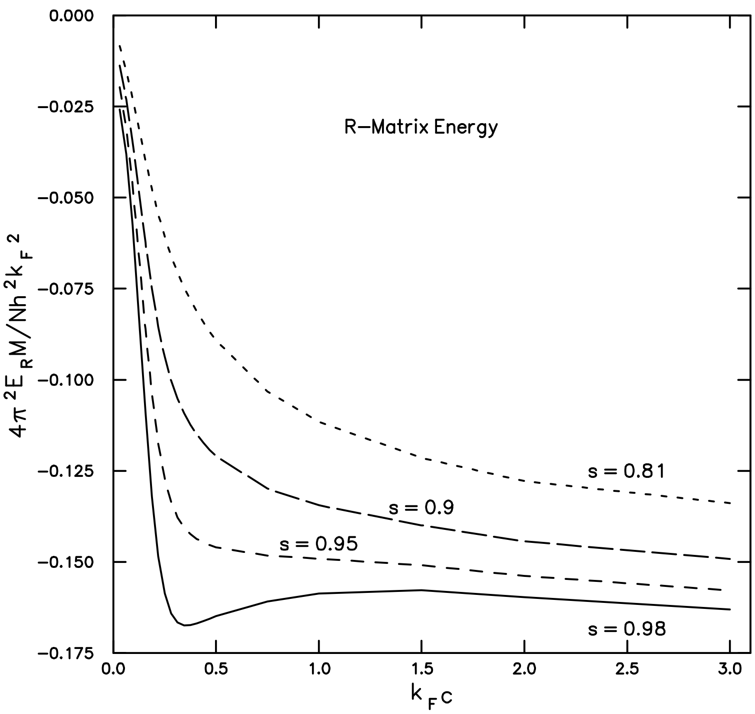

Figure 1. The numerical evaluation of the R-matrix energy. The short dashed curve is for $s = 0.81$, the long dashed curve is for $s = 0.9$, the dashed curve is for $s = 0.95$, and the solid curve is for $s = 0.98$.

and for our problem we want $s = 1$. The potential energy expansion is

$$\frac{E}{N} = \frac{3\hbar^2 k_F^2}{10M} + \frac{\pi^2\hbar^2}{4Mc^2}A_1 s + \frac{\pi^4\hbar^2}{16Mc^2}A_2 s^2 + \ldots,$$

The first term for neutrons is:

$$A_1 = \frac{3}{4\pi\kappa_F^3 V_0}\int_{|\vec{\mu}|\leq\kappa_F,\,|\vec{\nu}|\leq\kappa_F} d\vec{\mu}\,d\vec{\nu}\left[\tilde{v}(0) - \frac{1}{2}\tilde{v}(|\vec{\mu}-\vec{\nu}|)\right],$$

In terms of the dimensionless variables,

$$\vec{\rho} = \vec{r}/c, \quad \vec{\kappa} = c\vec{k} \quad c^3\tilde{v}(\kappa) = \frac{1}{(2\pi)^3}\int d\vec{r}\,V(r)\exp(-i\vec{k}\cdot\vec{r}).$$

$\tilde{v}(\kappa)$ can be worked out exactly and $\tilde{v}(0) = -1/(6\pi^2)$. Since we are concerned with the limit as $c \to 0$, and finite Fermi momentum, $\kappa_F \to 0$. Thus we get,

$$A_1 = -\frac{\pi}{3}\kappa_F^3.$$

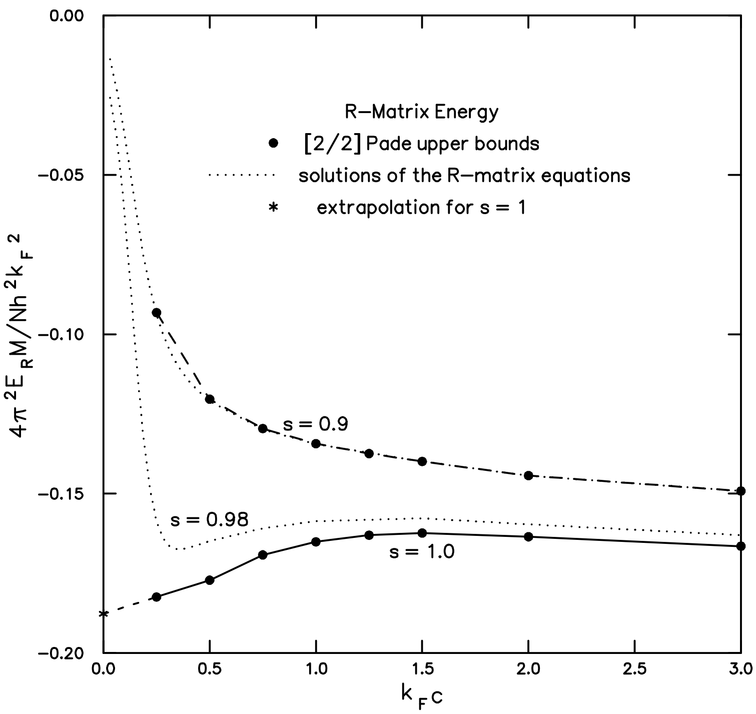

Figure 2. The Padé approximant upper bounds on the R-matrix-approximation energy divided by $\hbar^2 k_F^2/M$ for various potential strengths. Some of the numerical solutions of the R-matrix equation are included for reference.

For the time being, we will hold the scattering length fixed and finite, and let $c \to 0$. The potential strength stays finite, but the potential depth becomes infinite. The standard way to deal with this situation is to put ladder insertions in all the higher order terms. Skipping the details, we have for low-density

$$\frac{EM}{N\hbar^2} = k_F^2 \left[\frac{3}{10} + \frac{1}{3\pi} k_F a + 0.055661 (k_F a)^2 \right.$$
$$\left. + 0.00914 (k_F a)^3 - 0.018604 (k_F a)^4 + o(k_F^4) \right].$$

which just depends on the scattering length a and not on the shape of the potential. The case of interest is, of course, given by the limit as $a \to \infty$. Before considering the limit $a \to \infty$, our approach is to take some guidance from the low density expansion. Usually one would start with the K-matrix, however in the case of a purely attractive potential, it is plagued[3] with "Emery Singularities." Consequently, I will use the R-matrix[4] formulation. The difference between the K-matrix in ladder approximation and the R-matrix is in the Green's function. For the K-

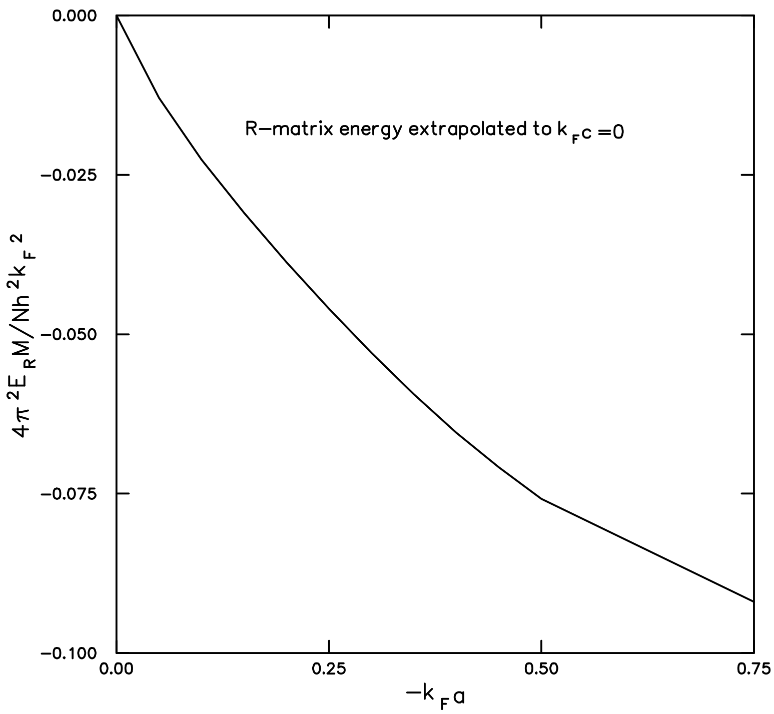

Figure 3. The extrapolation of the R-matrix energy to $k_F c = 0$ as a function of $k_F a$.

matrix the Green's function is

$$G_{k,l}(r,r') = \int_0^\infty \frac{k''^2 j_l(k''r) j_l(k''r')}{k''^2 - k^2} F(p,k'')\, dk'',$$

where F reflects the Pauli priniciple. It has been shown that,

$$K_l(k) = \frac{R_l(k)}{1 + (\frac{1}{2}\tau_1 - k^2/k_F) R_l(k)},$$

where

$$\begin{aligned}\tau_1 = &(k_F p)^{-1}\{(k^2 + \frac{1}{4}p^2 - k_F^2) \\ &\times \ln[(k_F^2 + k_F p + \frac{1}{4}p^2 - k^2)/(k_F^2 - \frac{1}{4}p^2 - k^2)] \\ &+ \left(1 - \frac{p^2}{4k_F^2}\right) \ln[(k_F + \frac{1}{2}p)/(k_F - \frac{1}{2}p)]\} \\ &+ \left(\frac{k}{k_F}\right) \ln[(k_F + \frac{1}{2}p + k)/(k_F + \frac{1}{2}p - k)].\end{aligned}$$

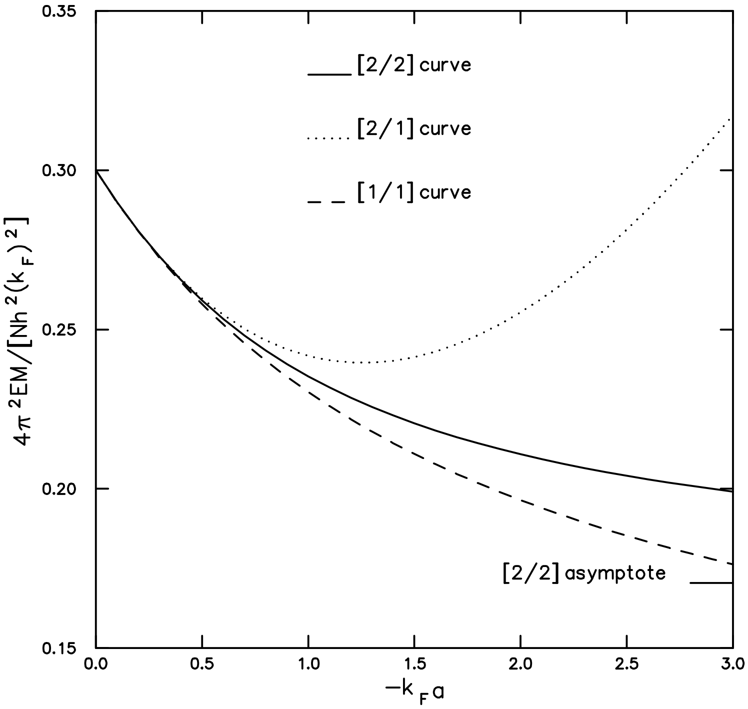

Figure 4. The ratio of the many-body energy per particle to $\hbar^2 k_F^2/M$, verses $-ak_F$. For the case of interest, $a \rightarrow -\infty$ is expected.

τ_1 is lower semi-bounded, but diverges logarithmically to plus infinity. However, when it is negative, there is a singularity in the K-matrix. One consequence of this result is that although it is expected that the radius of convergence of R in powers of the strength s is unity, the radius of convergence of the K-matrix series is zero.

In figure 1 we see the numerical results of the evaluation of the R-matrix energy. Notice that outside a small initial region, these curves are relatively flat.

2.1 Method 1

A series expansion in the potential strength can be computed numerically for the R-matrix. An examination of its structure shows it to be that of a two-side (or Hamburger) moment problem. It has been shown that for this case, inside the radius of convergence, that all Padé approximants[5] form upper bounds.

These numerical results are displayed in figure 2. The value sought, is the extrapolation to $k_F c = 0$ which is about $-0.18\hbar^2 k_F^2/M$.

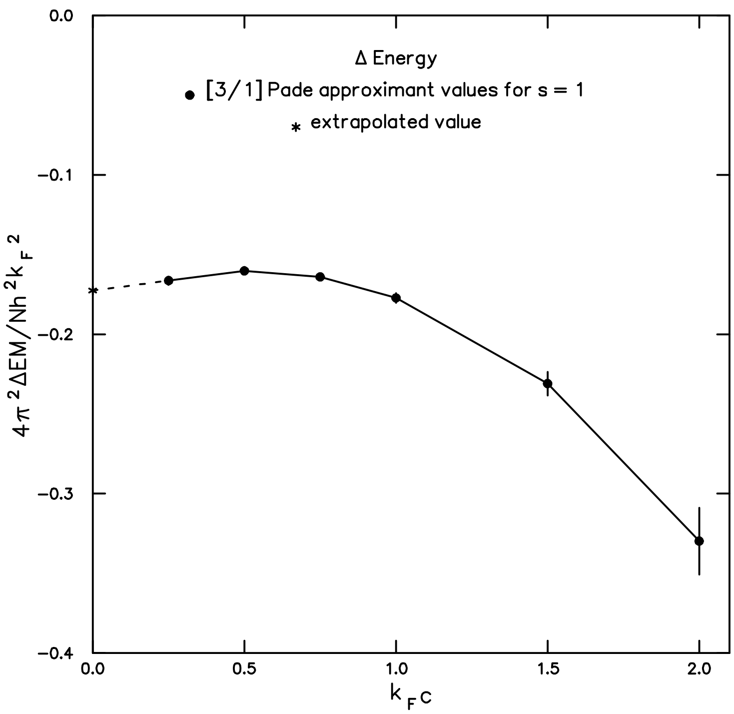

Figure 5. The estimates of the many-body energy per particle based on the series expansions in potential strength. The extrapolation to $k_F c = 0$ is also shown. The error bars reflect only the coefficient uncertainty.

2.2 Method 2

By adjusting the potential strength and k_F we can compute the behavior of the R-matrix energy as $k_F c \to 0$ for fixed $k_F a$. In figure 3 there is a plot of the results.

Here we need to extrapolate this curve to $k_F a = -\infty$. At low Fermi momentum the leading coefficient should be $1/(3\pi)$ so our extrapolation is about 2% low here. This behavior is not inconsistent with the results of the previous plot.

A bit of additional information is that the asymptote for the [2/2] Padé approximant to the ladder energy is about $0.24\hbar^2 k_F^2/M$ which is not vastly different from our estimates for the R-matrix energy, and also corresponds to no negative energy ground state.

We are now in a position to apply method 2 to the complete energy. We can compute various Padé approximants to the low density expansion, yielding the results displayed in figure 4.

Numerically, the asymptote for the [2/2] is $0.1705\hbar^2 k_F^2/M$. The value corresponds to a shift in the complete energy from the ideal gas energy of $\Delta E =$

$-0.1295\hbar^2 k_F^2/M$.

If we now apply method 1 to previously computed data the best Padé approximant is the [3/1]. These results are shown in Figure 5.

The result of this calculation is about $\Delta E = -0.17\hbar^2 k_F^2/M$. All together, I estimate that this model of the interactions in neutron matter gives $\Delta E = -(0.17 \pm 0.04)\hbar^2 k_F^2/M$.

3 Conclusions

The reasonable concordence of both methods for the computation of the ground-state energy means that the ground state of system behaves like that of a Fermi liquid, with an effective mass of $(2.3 \pm 0.5)M$. The wave-function is expected to correspond to that structure, aside from a set of exceptional points where $\vec{r}_i = \vec{r}_j$, the origins of the set of relative coordinates between all the pairs.

Acknowledgements

I wish to thank Prof. M. DeLlano for drawing ref. 6 to my attention. It contains a nice survey of the properties of the contact (delta function) interaction in various dimensions.

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